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Orbital angular momentum and massless particles

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Abstract. We analyse certain realizations of the rotation group in which half-integer representations appear, even though we employ scalar fields only. This angular momentum is interpreted as being purely orbital, and is shown to apply only to massless particles. We discuss the double-valued nature of these representations, including transformation properties of the spinors, which are formed only from space coordinates $\boldsymbol{x}$ and $\partial/\partial\boldsymbol{x}$. We describe also position and momentum operators which can be defined.

1. Introduction

The Dirac monopole provides an interesting example of how half-integral angular momentum can appear in a physical situation in which only scalar fields are present. The interpretation is that the non-local electromagnetic field carries the half-integral quantum numbers. The invariance group of the Dirac Hamiltonian was found by Fierz (1944) to be the rotation group, and integrability requirements (Hurst 1968) of these generators then lead to the Dirac charge quantization. More recent analyses have been given of these SO(3) representations (Frenkel and Hraskó 1976, Wu and Yang 1976), and include discussions of basis states and the global form of the rotations.

We present here a second example of how half-integral angular momentum can appear using only scalar fields. Again, the rotation group is involved, but the interpretation is very different. We find that our generators $\boldsymbol{J}$ can be written formally as $\boldsymbol{J}=\boldsymbol{Q}\times\boldsymbol{P}$, where $\boldsymbol{Q}$ and $\boldsymbol{P}$ satisfy the canonical commutation relations for position and momentum operators, so that we have purely orbital angular momentum. However, contrary to the usual situation, we find that this angular momentum can take half-integral values. We deduce then that these realizations apply to free massless particles. It will be seen that we are able to analyse the angular momentum properties of such a massless particle adequately in a rotation group context, but we do not discuss the linear momentum operators which, inevitably, have indeterminate matrix elements as well as Hermiticity difficulties. However, the angular momentum operators $\boldsymbol{J}$ are legitimate quantum mechanical operators, being Hermitian with normalizable eigenstates.

Let us explain the circumstances in which orbital angular momentum can take half-integral values. Let $\boldsymbol{J}=\boldsymbol{Q}\times\boldsymbol{P}$, where $\boldsymbol{Q}$ and $\boldsymbol{P}$ satisfy the canonical commutation relations

$$\begin{aligned}&[Q_i, Q_j]=0=[P_i, P_j]\\&[Q_i, P_j]=\mathrm{i}\delta_{ij},\end{aligned}\tag{1.1}$$

and let $J^2\psi=j(j+1)\psi$. Purely algebraic arguments (§ 3) indicate that two possibilities exist for the values of j, either $j=0, 1, 2, \ldots$, or $j=\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$. The conclusion that only integral values of j are allowed is usually obtained through a wave mechanical derivation, in which one uses the Schrödinger representation of operators satisfying (1.1). In view of the theorem of von Neumann concerning the uniqueness of the Schrödinger representation (a statement and discussion of this theorem is given by Reed and Simon 1972, chap. 8), it might be thought that the eigenvalue question is settled. However, von Neumann's theorem applies only to the groups generated by $\boldsymbol{Q}$ and $\boldsymbol{P}$, not to the fundamental commutation relations (1.1). More precisely, let

$$U(\boldsymbol{t})=\exp(\mathrm{i}\boldsymbol{t}\cdot\boldsymbol{P}), \qquad V(\boldsymbol{s})=\exp(\mathrm{i}\boldsymbol{s}\cdot\boldsymbol{Q}),$$

where $\boldsymbol{t}=(t_1, t_2, t_3)$ and $\boldsymbol{s}=(s_1, s_2, s_3)$ are three-component parameters. Then the Schrödinger representation is the unique (up to multiplicity and unitary equivalence) solution of the Weyl relations

$$U(\boldsymbol{t})V(\boldsymbol{s})=\exp(\mathrm{i}\boldsymbol{t}\cdot\boldsymbol{s})V(\boldsymbol{s})U(\boldsymbol{t}). \tag{1.2}$$

Thus there could be realizations of operators satisfying equations (1.1), for which the groups do not satisfy the Weyl relations and so are inequivalent to the Schrödinger representation. Indeed, such examples are known (Reed and Simon 1972). The eigenvalue arguments which we employ are independent of the representation used, and we find that the restriction on j, imposed by the special form of the orbital angular momentum operators, is that j may vary only by integral steps, leaving the two possibilities mentioned above. Because these two spectra are different it is immediately clear that the position and momentum operators $\boldsymbol{Q}$, $\boldsymbol{P}$ which describe the latter case are inequivalent to the corresponding operators of the Schrödinger representation.

We can see immediately that any particle with mass must take only the integral values of j, and so can be described in the usual way with the Schrödinger representation. However, the particle with $j=\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$ must be massless. For, consider a (massive) particle in its rest frame, that is, consider only the states ψ such that $\boldsymbol{P}\psi=0$. Using

$$J^2=Q^2P^2+\mathrm{i}\boldsymbol{Q}\cdot\boldsymbol{P}(\mathrm{i}\boldsymbol{Q}\cdot\boldsymbol{P}+1), \tag{1.3}$$

it follows that $J^2\psi=0$, so that any massive particle contains within its orbital angular momentum spectrum the value $j=0$, and consequently also $j=1, 2, 3, \ldots$. The particle described by half-integral j has a lowest state of $j=\frac{1}{2}$, and therefore cannot fall into the state with $j=0$. According to the argument above, such a particle has no rest frame and so must be massless. Evidently, its position and momentum vectors can never be parallel. We emphasize that the interpretation of the generators $\boldsymbol{J}$ which we present, as the orbital angular momentum operators for massless particles is based on the decomposition $\boldsymbol{J}=\boldsymbol{Q}\times\boldsymbol{P}$, which is of a formal nature only; we are interested here mainly in the properties of $\boldsymbol{J}$, as a means of introducing spinor representations using only scalar fields. The methods used here are applicable to a relativistic treatment, and it is hoped to complete this work in the future.

In § 2 we discuss the Schrödinger representation and formulate the usual orbital angular momentum theory in a group theoretical manner consistent with the approach to be used later. We examine the eigenvalue question and conclude that only integral values of orbital angular momentum are allowed. In § 3 we present representation-independent arguments on possible values of j, and find that j must vary by integral steps.

Next, we construct those representations of SO(3) which are labelled by half integers, the spinor (double-valued) representations. Here we use some recent techniques (Lohe and Hurst 1971) developed for the study of the orthogonal groups, the most important of which is the use of operators a_i, 'traceless boson operators', with properties specially tailored for use in SO(3). These operators (defined by equation (4.1) below), enable one to project easily from the space of functions $\psi(\boldsymbol{x})$ onto the subspace of harmonic functions $\psi(\boldsymbol{a})$. Operators and functions in $\boldsymbol{x}$ and $\partial/\partial\boldsymbol{x}$ can be written more concisely and in a more transparent form using only $\boldsymbol{a}$ and $\partial/\partial\boldsymbol{x}$. The advantage of this approach has been demonstrated (Lohe and Hurst 1971) in applications to the orthogonal groups, particularly in the discussion of Weyl tensors and orthonormal basis states, where general results can be obtained with methods analogous to those employed for the unitary groups. However, there is an additional, remarkable property of $\boldsymbol{a}$ which is not possessed by $\boldsymbol{x}$, and which is the cornerstone of our construction: the operators a_i can be made to transform as either vector or spinor components as required. In this way one does not need to introduce new spinorial objects, but can construct the double-valued representations by employing as a representation space the functions on the unit sphere. Here, we mean by 'representation' a homomorphism that could be double-valued, and so we allow for a representation T_g, $g \in \mathrm{SO}(3)$, that

$$T_{g_1}T_{g_2} = \pm T_{g_1 g_2}, \tag{1.4}$$

where the sign cannot be uniquely chosen. The necessity to permit such double-valued representations in quantum mechanics has been explained by Wigner (1939).

The relevance of the doubly-connected nature of the SO(3) manifold to properties of the real world can be demonstrated in a striking way by a construction due to Dirac, termed 'the spinor spanner' (Bolker 1973). In this demonstration one carries out certain operations on a bundle of strings (including rotations), for which a rotation by 4π and not by 2π is equivalent to the identity rotation. A description is given also by Gardner (1966) and Misner *et al* (1973).

The problem of constructing SO(3) spinor representations in the space of functions on the unit sphere has been solved (Lohe 1973) in the general context of the orthogonal groups. We describe the ingredients of this construction in § 4, using here the familiar form of metric encountered in physical situations, and clarifying the role of $\boldsymbol{a}$ as a spinor operator. Also, it is verified that the representations are double-valued by computing the representation matrix elements, which span the space of functions on the group manifold, and which turn out to be the Wigner functions $D^J_{mm'}(\alpha, \beta, \gamma)$. It will be seen that we can utilize the same basis functions, the solid spherical harmonics, for the construction of both spinor and tensor representations by employing different forms of the generators. These different forms (equation (4.11) below) are labelled by a parameter λ taking values $\lambda = 0, \frac{1}{2}, 1, \ldots$; integral values of λ generate the tensor representations and half-integral values generate the spinor representations. It will appear (§ 5) that there are actually only two distinct cases, which we take to be either $\lambda = 0$, giving the usual theory of angular momentum, or $\lambda = \frac{1}{2}$, leading to the new results.

Finally, in § 5 we describe an equivalence property which can be extrapolated to provide a method of defining position and momentum operators. This leads to the interpretation of $\boldsymbol{J}$ as orbital angular momentum. We also show that $\boldsymbol{J}$ is an Hermitian operator.

Some calculations are detailed in the appendixes.

2. The Schrödinger representation

In the Schrödinger representation the orbital angular momentum takes the form

$$\boldsymbol{L} = -\mathrm{i}\boldsymbol{x} \times \boldsymbol{\partial}, \tag{2.1}$$

(where $\boldsymbol{\partial} = \partial/\partial \boldsymbol{x}$) and satisfies

$$[L_i, L_j] = \mathrm{i}\epsilon_{ijp}L_p \qquad \text{(summation)}. \tag{2.2}$$

It is usual to transform to spherical polar coordinates†,

$$\begin{aligned} x_\pm &= \mathrm{e}^{\pm\mathrm{i}\phi} r \sin\theta \\ x_3 &= r\cos\theta, \end{aligned} \tag{2.3}$$

and to solve the simultaneous differential equations

$$\begin{aligned} L^2\psi &= l(l+1)\psi, \\ L_3\psi &= m\psi. \end{aligned} \tag{2.4}$$

The angular dependence of the particle wavefunctions is then the spherical harmonics $Y_{lm}(\theta, \phi)$, and l takes integral values 0, 1, 2, . . . , with the magnetic quantum number m satisfying $|m| \leqslant l$ (showing also that the dimension of the representation is $2l+1$).

In the group theoretic approach one starts by defining a representation T_g of the group SO(3) according to

$$T_g\psi(\boldsymbol{x}) = \psi(\boldsymbol{x}g). \tag{2.5}$$

Here $\boldsymbol{x}$ is a row vector, and g is a 3×3 matrix belonging to SO(3). The group generators, calculated by standard techniques (see for example Miller 1968) are the operators $\boldsymbol{L}$ given by equation (2.1). The representation will be irreducible if $\psi(\boldsymbol{x})$ is a harmonic homogeneous polynomial in $\boldsymbol{x}$ of degree l, that is, if $\psi(\boldsymbol{x})$ satisfies

$$\begin{aligned} \Delta\psi &= 0, \\ N\psi &= l\psi, \end{aligned} \tag{2.6}$$

where $\Delta = \boldsymbol{\partial}\cdot\boldsymbol{\partial}$ and $N = \boldsymbol{x}\cdot\boldsymbol{\partial}$ are the Laplace and Euler operators respectively. Next, one uses the theorem of Cartan, showing that an irreducible representation space contains a unique highest-weight polynomial (a proof is given by Zhelobenko 1962). This polynomial is found to be

$$\psi_{ll} = x_+^l. \tag{2.7}$$

Then, by applying the lowering operator L_- one obtains the basis states as Gegenbauer polynomials:

$$\psi_{lm}(\boldsymbol{x}) = r^{l-m}x_+^m C_{l-m}^{m+\frac{1}{2}}(x_3/r). \tag{2.8}$$

In polar coordinates these states are just the solid spherical harmonics $r^l Y_{lm}(\theta, \phi)$.

As we have mentioned, the construction outlined so far leads only to the representations of SO(3) for which the label l is integral, and we now examine the reasons for this restriction. Let us take for example the case $l = \frac{1}{2}$, for which the representation is two

† We use the notation, given a three-vector $\boldsymbol{A}$, that $A_\pm = A_1 \pm \mathrm{i}A_2$.

dimensional, and try to repeat the previous steps. One has a state of highest weight which is (from equation (2.7)):

$$\psi_{1/2,1/2}=(x_+)^{1/2}, \tag{2.9}$$

and by applying L_- one obtains $\psi_{1/2,-1/2}$:

$$\psi_{1/2,-1/2}=x_3(x_+)^{-1/2}.$$

It is here that the attempted construction fails, because $L_-\psi_{1/2,-1/2}$ is not zero as it should be, and in fact one can continue applying L_- indefinitely without reaching the zero vector. One might hope then that a suitable set of basis states is provided by the spherical harmonics $Y_{lm}(\theta,\phi)$, defined for half-integral l and m via equation (2.8). However, such states do not transform linearly amongst themselves under SO(3) transformations (as the above example shows), and so the attempt again fails.

There remains a puzzle concerning this failure. From the point of view of representation theory one is using two well known theorems, that all unitary irreducible representations of compact groups are finite dimensional (Zhelobenko 1973), and that each irreducible representation space is the cyclic envelope of a unique state, the highest-weight state (Zhelobenko 1962). Now, according to a theorem of Weyl (1946), one also knows that all representations of compact groups are completely reducible; one can therefore construct an irreducible representation of SO(3) by choosing a state of highest weight and applying the lowering operator L_-, and expect to generate a finite-dimensional set of basis states. In practice we have found, choosing the state of highest weight as in equation (2.9), that an infinite-dimensional space has been obtained, and that an indecomposable representation has in fact been constructed. Apparently one has encountered representations of the Lie algebra which do not integrate to unitary finite-dimensional representations of the group. We will verify explicitly in § 4 that the double-valued representations of SO(3) which we construct are in fact finite dimensional, and unitarity can then be imposed.

3. General eigenvalue arguments

It is necessary to search for general arguments on possible eigenvalues of J^2, since, as explained in § 1, there might exist representations of $\boldsymbol{Q}$ and $\boldsymbol{P}$ inequivalent to the Schrödinger representation which allow J^2 to take a different spectrum. In order to obtain representation-independent results, we use only the commutation relations of $\boldsymbol{Q}$ and $\boldsymbol{P}$ (equations (1.1)) which are to act on some abstract, unspecified Hilbert space.

A suitable argument is due to Green (1965), and may be stated as follows. Let ψ_j be an eigenvector of J^2 with eigenvalue $j(j+1)$, so that $J^2\psi_j=j(j+1)\psi_j$. Then

$$\begin{aligned} J^2\phi_j^{(-)}&=j(j-1)\phi_j^{(-)},\\ J^2\phi_j^{(+)}&=(j+1)(j+2)\phi_j^{(+)}, \end{aligned} \tag{3.1}$$

where

$$\phi_j^{(-)}=(\boldsymbol{Q}\cdot\boldsymbol{\sigma})(\boldsymbol{J}\cdot\boldsymbol{\sigma}-j)\psi_j,$$

and

$$\phi_j^{(+)}=(\boldsymbol{Q}\cdot\boldsymbol{\sigma})(\boldsymbol{J}\cdot\boldsymbol{\sigma}+j+1)\psi_j.$$

Here $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ are the usual 2×2 Pauli matrices. The proof of equation (3.1) involves only the evaluation of commutators and is found in Green (1965). Equations (3.1) state that for a given value j of angular momentum the values $j-1$ and $j+1$ are also allowed. Hence j varies only by integral amounts, thereby eliminating all but two possible spectra, either $j = 0, 1, 2, \ldots,$ or $j = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$.

Now there is a further condition to be satisfied, which is that J^2 can have only non-negative eigenvalues. One sees from equation (3.1) that this condition can be fulfilled for $j = \frac{1}{2}$ only if $\phi^{(-)}_{1/2} = 0$, that is, if

$$\phi^{(-)}_{1/2} = \boldsymbol{Q}\cdot\boldsymbol{\sigma}(\boldsymbol{J}\cdot\boldsymbol{\sigma} - \tfrac{1}{2})\psi_{1/2} = 0. \tag{3.2}$$

This is a necessary condition for the existence of the half-integral spectrum, and is not satisfied in the case of the Schrödinger representation. Let us also point out that similarly $\phi^{(-)}_j$ must be zero at $j = 0$, in the case of integral eigenvalues, and that the Schrödinger representation indeed satisfies this condition.

We can present these results in a more systematic way, and without the use of Pauli matrices. Firstly, we need to find the selection rules on matrix elements of $\boldsymbol{Q}$ and $\boldsymbol{P}$, that is, we determine the values of $\Delta j = j' - j$ for which $(\psi_{j'}, \boldsymbol{Q}\psi_j)$ is non-zero. Here $(\cdot,\cdot)$ denotes the inner product of the Hilbert space on which $\boldsymbol{Q}$, $\boldsymbol{P}$ act as Hermitian operators. We use the fact that $\boldsymbol{Q}$ (and similarly $\boldsymbol{P}$) are vector operators under SO(3) transformations:

$$[J_i, Q_j] = \mathrm{i}\epsilon_{ijp}Q_p. \tag{3.3}$$

The following result is then obtained using equation (3.3) and the fact that $\boldsymbol{Q}\cdot\boldsymbol{J} = 0$:

$$[J^2, [J^2, \boldsymbol{Q}]] = 2J^2\boldsymbol{Q} + 2\boldsymbol{Q}J^2. \tag{3.4}$$

Taking matrix elements of both sides leads to the equation

$$(j - j' + 1)(j - j' - 1)(j + j')(j + j' + 2)(\psi_{j'}, \boldsymbol{Q}\psi_j) = 0,$$

showing that $(\psi_{j'}, \boldsymbol{Q}\psi_j) = 0$ unless $\Delta j = j' - j = \pm1$. For the special case $j = j' = 0$ the matrix element is also zero, as is shown by the explicit decomposition of $\boldsymbol{Q}$ given below. The same selection rules hold for $\boldsymbol{P}$ also.

These selection rules indicate that we can decompose $\boldsymbol{Q}$ into two vector parts, $\boldsymbol{Q}^{(+)}$ and $\boldsymbol{Q}^{(-)}$, such that

$$\boldsymbol{Q} = \boldsymbol{Q}^{(+)} + \boldsymbol{Q}^{(-)},$$

and with the property that for $\boldsymbol{Q}^{(+)}$ we have $\Delta j = +1$, and for $\boldsymbol{Q}^{(-)}$ we have $\Delta j = -1$. The operators $\boldsymbol{Q}^{(\pm)}$ will behave as raising and lowering operators with respect to the label j, and one is then able to write the eigenvectors $\phi^{(\pm)}_j$ appearing in equation (3.2) as $\phi^{(\pm)}_j = \boldsymbol{Q}^{(\pm)}\psi_j$, and also as $\phi^{(\pm)}_j = \boldsymbol{P}^{(\pm)}\psi_j$. Decompositions of this kind for vector operators have been studied by Bracken and Green (1971) and suitable projection operators obtained as functions of $\boldsymbol{J}$. In the no ation of their paper, to which we refer for details, the generators are $\alpha_{kl} = \mathrm{i}\epsilon_{klp}J_p$. Let us note that an arbitrary SO(3) vector $\boldsymbol{A}$ will normally be decomposed into three vector parts,

$$\boldsymbol{A} = \boldsymbol{A}^{(+)} + \boldsymbol{A}^{0} + \boldsymbol{A}^{(-)},$$

where for $\boldsymbol{A}^0$ one has $\Delta j = 0$, but that the vectors $\boldsymbol{Q}$ and $\boldsymbol{P}$ do not have components $\boldsymbol{Q}^0$ and $\boldsymbol{P}^0$ by virtue of the special property $\boldsymbol{Q}\cdot\boldsymbol{J} = 0 = \boldsymbol{P}\cdot\boldsymbol{J}$. In our problem, $\boldsymbol{Q}$ and $\boldsymbol{P}$ (and combinations from them) are the most general vectors at hand.

The characteristic identity for SO(3), which indicates how one may define the necessary projection operators, reads

$$(\alpha-\Lambda-1)(\alpha+\Lambda)(\alpha-1)=0. \tag{3.5}$$

Here α stands for the matrix with elements α_{ij}, and Λ is the implicit operator defined according to $J^2=\Lambda(\Lambda+1)$, and as such has eigenvalues j. Now, following Bracken and Green (1971) one obtains

$$\boldsymbol{Q}^{(-)}=(\Lambda+1)\boldsymbol{Q}+\mathrm{i}\boldsymbol{J}\times\boldsymbol{Q}, \tag{3.6a}$$

$$\boldsymbol{Q}^{(+)}=\Lambda\boldsymbol{Q}-\mathrm{i}\boldsymbol{J}\times\boldsymbol{Q}. \tag{3.6b}$$

(Some details of the calculation are included in appendix 1.) Thus, we have that equations (3.1) hold with $\phi_j^{(\pm)}=\boldsymbol{Q}^{(\pm)}\psi_j$, and also with $\phi_j^{(\pm)}=\boldsymbol{P}^{(\pm)}\psi_j$. Here $\boldsymbol{P}^{(\pm)}$ is defined analogously to $\boldsymbol{Q}^{(\pm)}$:

$$\boldsymbol{P}^{(-)}=(\Lambda+1)\boldsymbol{P}+\mathrm{i}\boldsymbol{J}\times\boldsymbol{P},$$

$$\boldsymbol{P}^{(+)}=\Lambda\boldsymbol{P}-\mathrm{i}\boldsymbol{J}\times\boldsymbol{P}.$$

(If desired, one could verify equations (3.1) directly using these definitions of $\boldsymbol{Q}^{(\pm)}$ and $\boldsymbol{P}^{(\pm)}$ without recourse to the characteristic identity (3.5).)

The necessary condition (3.2) for the admission of half-integral j, namely $\phi_{1/2}^{(-)}=0$, now imposes the following conditions:

$$\phi_{1/2}^{(-)}=0=\boldsymbol{Q}^{(-)}\psi_{1/2}=\boldsymbol{P}^{(-)}\psi_{1/2}. \tag{3.7}$$

There are actually twelve conditions to be satisfied here, since there are two independent states $\psi_{1/2}$ in the two-dimensional representation $j=\frac{1}{2}$. It remains to be seen whether it is in fact possible to construct operators $\boldsymbol{Q}$ and $\boldsymbol{P}$ which satisfy these conditions, but this will be done in appendix 2, and equations (3.7) explicitly verified. Thus the arguments of this section are the most general that can be applied to the problem. Finally, we remark that it is the selection rules of $\boldsymbol{Q}$ and $\boldsymbol{P}$, namely, the fact that $\Delta j=\pm 1$ rather than $\Delta j=\pm\frac{1}{2}$, which force j to vary only by integral rather than half-integral amounts.

4. Spinor representations

We now wish to find a construction of the spinor representations of SO(3) using the functions on the sphere as basis states, without introducing new spinorial objects; these representations must be double-valued (as will be explicitly verified). The method to be used has been described by Lohe (1973), and we refer to this paper for motivation of this construction and further details. It is necessary to recognize the fundamental role of the following operators:

$$\boldsymbol{a}=\boldsymbol{x}-r^2\boldsymbol{\partial}(2N+1)^{-1}. \tag{4.1}$$

These operators, traceless bosons, are the harmonic projections of the vector components $\boldsymbol{x}_i$ in the following sense. Let R^l denote the space of homogeneous polynomials of degree l, and H^l the subspace of harmonic polynomials. Then, one has the direct sum decomposition (Vilenkin 1968)

$$R^l=H^l\oplus r^2R^{l-2},$$

which implies a corresponding decomposition for $\boldsymbol{x}$:

$$\boldsymbol{x} = \boldsymbol{a} + r^2(4N+6)^{-1}\Delta\boldsymbol{x},$$

where

$$\boldsymbol{a} = [1 - r^2(4N+6)^{-1}\Delta]\boldsymbol{x} = \boldsymbol{x} - r^2\boldsymbol{\partial}(2N+1)^{-1}$$

in H^l. This expression is determined by the requirement that $\Delta a_i h(\boldsymbol{x}) = 0$, in order that the range of $\boldsymbol{a}$ lie in H^l. The following commutation relations are satisfied by $\boldsymbol{a}$:

$$[a_i, a_j] = 0, \tag{4.2}$$

$$[\partial_i, a_j] = \delta_{ij} - 2a_i\partial_j(2N+1)^{-1}. \tag{4.3}$$

An important property is the following traceless condition, for it shows that tensors constructed from a_i are automatically traceless:

$$a^2h(\boldsymbol{x}) = (a_+a_- + a_3^2)h(\boldsymbol{x}) = 0, \tag{4.4}$$

where $h(\boldsymbol{x})$ is a harmonic function. Thus the use of a_i enables one to perform an easy construction of basis states for irreducible representations of SO(3), because an arbitrary harmonic polynomial can be built up by repeated applications of a_i to a reference state, which is taken to be the constant 1. If $h(\boldsymbol{x})$ is a harmonic polynomial, it can be written more simply, by virtue of equation (4.4), as a polynomial in $\boldsymbol{a}$, $\psi(\boldsymbol{a})$. (It is understood that the operators $\boldsymbol{a}$ in such a polynomial are acting to the right on the constant 1.)

We can now reformulate the representation theory of SO(3), as described in § 2, in terms of $\boldsymbol{a}$ and $\boldsymbol{\partial}$. The irreducible representation space consists of homogeneous polynomials ψ in $\boldsymbol{a}$, and T_g is defined by:

$$T_g\psi(\boldsymbol{a}) = \psi(\boldsymbol{a}g). \tag{4.5}$$

The generators are

$$\boldsymbol{L} = -\mathrm{i}\boldsymbol{a} \times \boldsymbol{\partial} = -\mathrm{i}\boldsymbol{x} \times \boldsymbol{\partial} \tag{4.6}$$

as before, using equation (4.1). It is evident that under these transformations $\boldsymbol{a}$ behaves as a vector operator:

$$[L_i, a_j] = \mathrm{i}\epsilon_{ijp}a_p. \tag{4.7}$$

The basis states are simply (Lohe and Hurst 1971)

$$\psi_{lm}(\boldsymbol{a}) = (-1)^{l-m}a_+^m a_3^{l-m}, \tag{4.8}$$

$|m| \leqslant l$, and are necessarily equal to the states (2.8) previously derived, and so are actually the solid spherical harmonics. This example shows how a harmonic polynomial takes a simpler and more workable form when expressed as a polynomial in $\boldsymbol{a}$.

It is well known that in a unitary representation the matrix elements of $L_\pm$ are:

$$L_\pm\psi_{lm} = [(l \mp m)(l \pm m + 1)]^{1/2}\psi_{l,m\pm1}.$$

From this the inner product, which need be defined only for basis states, is then determined to be:

$$\langle\psi_{l'm'}, \psi_{lm}\rangle = \delta_{ll'}\delta_{mm'}\frac{2^l l!(l-m)!(l+m)!}{(2l)!}. \tag{4.9}$$

To this point we have only re-formulated the representation theory described in § 2, but now we generalize the definition of the representation (4.5) to the following:

$$T_g\psi(\boldsymbol{a}) = \left(\frac{a_+}{(\boldsymbol{a}g)_+}\right)^{\lambda} \psi(\boldsymbol{a}g). \tag{4.10}$$

Thus the special case (4.5) is obtained by putting $\lambda = 0$. This multiplier representation was previously introduced (Lohe 1973), and required properties deduced in a general context, but for clarity we will derive the necessary results here directly.

Now, firstly it is readily checked that T_g is actually a representation: $T_{g_1g_2} = T_{g_1}T_{g_2}$. We also assert that $T_g\psi(\boldsymbol{a})$ will be a harmonic polynomial if ψ is, provided λ is a non-negative integer, that is, the range of the operator T_g is spanned by the set (or a subset) of the polynomials (4.8). This can be seen by first calculating the generators $\boldsymbol{J}$, which turn out to be:

$$\begin{aligned} J_+ &= L_+, \\ J_- &= L_- + 2\lambda\frac{a_3}{a_+}, \\ J_3 &= L_3 - \lambda, \end{aligned} \tag{4.11}$$

where $\boldsymbol{L}$ is given by (4.6), or explicitly,

$$\begin{aligned} L_+ &= a_3\partial_+ - a_+\partial_3, \\ L_- &= a_-\partial_3 - a_3\partial_-, \\ L_3 &= \tfrac{1}{2}(a_+\partial_- - a_-\partial_+). \end{aligned} \tag{4.12}$$

The commutation relations are readily checked:

$$[J_i, J_j] = \mathrm{i}\epsilon_{ijp}J_p. \tag{4.13}$$

Since $J_+ = L_+$, the state of highest weight is unchanged, and so is therefore a_+^l for some integer l, the polynomial degree. The representation label j is then deduced by finding the eigenvalue of J_3 on this state (denoted ψ_{jj}), and from (4.11) one obtains

$$j = l - \lambda. \tag{4.14}$$

Hence,

$$\psi_{jj} = a_+^{j+\lambda}. \tag{4.15}$$

The general state ψ_{jm} is now found by repeated applications of J_- to ψ_{jj}, and calculations show that

$$\psi_{jm} = (-1)^{j-m} a_+^{m+\lambda} a_3^{j-m}, \tag{4.16}$$

where $|m| \leqslant j$. It is possible now to see why the restriction of λ to non-negative integers ensures that only polynomials are obtained, because the representation space is spanned by the $2j+1 = 2l - 2\lambda + 1$ states (4.16), and these are a subset of the $2l+1$ harmonic functions (4.8) for integral $\lambda \geqslant 0$. Indeed, one readily verifies that $J_-\psi_{j,-j} = 0$, so that the representation cuts off as required at the minimum weight state $\psi_{j,-j}$. The meaning of the inverse operator in equation (4.10), and therefore also in (4.11), is now clear; we have shown that both the domain and range of the operators T_g are the set of functions (4.16), for which the inverse has only a symbolic meaning.

The representations defined by (4.10) are not unitary in the previous inner product (4.9), but, in the same way as before, a re-definition to give unitary representations can be made, and is found to be:

$$\langle \psi_{jm}, \psi_{j'm'} \rangle_\lambda = \delta_{jj'}\delta_{mm'} \frac{(j-m)!(j+m)!}{(2j-1)!!}. \tag{4.17}$$

The normalized states are therefore

$$\phi_{jm} = \left(\frac{(2j-1)!!}{(j-m)!(j+m)!}\right)^{1/2} \psi_{jm}. \tag{4.18}$$

Up to this point the introduction of the multiplier representation (4.10) has not led to new results, but has shown how the solid spherical harmonics of degree l can be made to carry any one of the SO(3) representations labelled by $l, l-1, \ldots, 1, 0$ depending on how λ is chosen. For this reason, the multipliers of (4.10) used in this way are mathematically trivial, and indeed do not lead to any new physical description of angular momentum.

However, there is an unexpected property which exposes a completely new structure, that of the manner by which the doubly-connected nature of the SO(3) group manifold expresses itself through its spinor representations. This property is that the multiplier in (4.10) can be written as a perfect square:

$$(\boldsymbol{a}g)_+/a_+ = \mathfrak{S}^2(\boldsymbol{a}, g), \tag{4.19}$$

where $\mathfrak{S}$ is a rational function in $\boldsymbol{a}$. This is most easily verified by using the Euler angles α, β, γ, rather than the parametrization of $g \in \mathrm{SO}(3)$ in terms of the elements g_{ij}. The definition of α, β, γ (chosen to agree with Wigner 1959) is:

$$\begin{aligned} g_{11}+\mathrm{i}g_{12}-\mathrm{i}g_{21}+g_{22} &= \mathrm{e}^{\mathrm{i}(\alpha+\gamma)}(\cos\beta+1), \\ g_{11}+\mathrm{i}g_{12}+\mathrm{i}g_{21}-g_{22} &= \mathrm{e}^{\mathrm{i}(\alpha-\gamma)}(\cos\beta-1), \\ g_{31}+\mathrm{i}g_{32} &= -\mathrm{e}^{\mathrm{i}\alpha}\sin\beta, \\ g_{13}+\mathrm{i}g_{23} &= \mathrm{e}^{-\mathrm{i}\gamma}\sin\beta, \\ g_{33} &= \cos\beta. \end{aligned} \tag{4.20}$$

Then we obtain (using (4.4)):

$$\mathfrak{S} = \mathrm{e}^{\frac{1}{2}\mathrm{i}(\alpha+\gamma)}\cos\tfrac{1}{2}\beta - \mathrm{e}^{\frac{1}{2}\mathrm{i}(\alpha-\gamma)}\sin\tfrac{1}{2}\beta(a_3/a_+). \tag{4.21}$$

This means that now λ can take half-integral values, and the results derived for integral λ apply also to half-integral values. This includes the form of the generators (4.11), and the labelling (4.14), the basis states (4.16) and the inner product (4.17). The labels j, m and λ are simultaneously all integral, or all half-integral. In taking the square root one has introduced double-valued representations, as is apparent from the multiplier (4.21), because $\mathfrak{S}$ changes sign under any one of the following transformations (comprising one complete rotation):

$$\begin{aligned} &\alpha \to \alpha + 2\pi, \\ &\beta \to \beta + \pi, \\ &\gamma \to \gamma + 2\pi. \end{aligned}$$

For half-integral j (and corresponding half-integral λ and m) the basis states are still polynomials and are therefore the solid spherical harmonics, which span the representation space of dimension $2j+1=2l-2\lambda+1$, for any choice of $\lambda=0,\frac{1}{2},1,\frac{3}{2},\ldots,l$. Thus the basis states are still single-valued despite the appearance of double-valued representations.

The difference in our approach compared to the unsuccessful attempt of § 2 is that we have taken a square root (4.19) in a way that retains the polynomial structure, unlike the square root of equation (2.9). As a result, while previously we tried to use the same generators (2.1) and different basis states for half-integral representations, now we use different generators (4.11) but the same basis states. An indecomposable infinite-dimensional representation is avoided, because finite-dimensional representation spaces are guaranteed from the finite dimensionality of the space of homogeneous polynomials of some fixed degree.

The origin of the multiplier (4.19) and its perfect square property is found in Lohe (1973) and Lohe and Hurst (1973), where the connection between the methods used here and those of Zhelobenko (1962, 1973) are demonstrated. In this latter method, which is suited to a discussion of the orthogonal groups from the mathematical viewpoint, representations are constructed on homogeneous spaces defined from a triangular decomposition of the group. The multiplier (4.19) appears through the method of induced representations, as a matrix minor $\Delta(zg)$, and plays a central role in this context. The property (4.19) is proved by Zhelobenko (1962) for general orthogonal groups, and results from a theorem suggested by Godement (1952) which ensures that the method constructs all representations. The method in general, which relies on the existence of a highest (dominant) weight state, is known as the Borel–Weil theorem†, but can be traced ultimately to the work of Cartan.

The essential role of the traceless bosons a_i is apparent in the fundamental spinor representation, labelled $j=\frac{1}{2}$. The normalized basis states are, from (4.16) and (4.18) (putting $\lambda=\frac{1}{2}$):

$$\begin{aligned}\phi_{1/2,1/2}&=a_+/\sqrt{2},\\ \phi_{1/2,-1/2}&=-a_3/\sqrt{2}.\end{aligned}\tag{4.22}$$

(One may check that $J_-\phi_{1/2,-1/2}=0$.) The matrix elements of T_g, obtained through equation(4.10), are given by:

$$T_g=\begin{pmatrix}\cos\frac{1}{2}\beta\, e^{-\frac{1}{2}i(\alpha+\gamma)} & -\sin\frac{1}{2}\beta\, e^{-\frac{1}{2}i(\alpha-\gamma)}\\ \sin\frac{1}{2}\beta\, e^{\frac{1}{2}i(\alpha-\gamma)} & \cos\frac{1}{2}\beta\, e^{\frac{1}{2}i(\alpha+\gamma)}\end{pmatrix}\tag{4.23}$$

which we recognize as the Wigner function $D^{1/2}(\alpha,\beta,\gamma)$. It is clear now that the two-component object $(a_+,-a_3)$ is behaving as a spinor, since it transforms like the fundamental spinor representation, despite the introduction of $\boldsymbol{a}$ as a vector operator. Thus the operators a_i, which are constructed only from space coordinates $\boldsymbol{x}$ and $\boldsymbol{\partial}$, serve as both vector and spinor components. Evidently, the spinor $(a_+,-a_3)$ changes sign under a rotation of 2π about any axis, and in this way exhibits its double-valued nature; nevertheless, when acting in the space of harmonic polynomials this spinor produces single-valued functions of $\boldsymbol{x}$.

Since the introduction of new objects behaving as spinors has been eliminated, our approach is distinct from the usual construction, which employs Cartan spinors (Cartan

† Borel A and Weil A 1954 *Seminaire Bourbaki* (exposé by J-P Serre).

1967). To clarify this difference, let us demonstrate how spinors arise in the usual formalism. One may start with a two-component Schwinger spinor (α_1, α_2) (Schwinger 1965), where α_1, α_2 are ordinary boson creation operators, with adjoints $\bar{\alpha}_1, \bar{\alpha}_2$, and construct the operators.

$$\begin{aligned} a'_+ &= -\alpha_1^2, \\ a'_- &= \alpha_2^2, \\ a'_3 &= \alpha_1\alpha_2. \end{aligned} \tag{4.24}$$

These operators behave as SO(3) vectors (i.e. satisfy equation (4.7)) provided the generators are defined as:

$$\begin{aligned} L'_+ &= \alpha_1\bar{\alpha}_2, \\ L'_- &= \alpha_2\bar{\alpha}_1, \\ L'_3 &= \tfrac{1}{2}(\alpha_1\bar{\alpha}_1 - \alpha_2\bar{\alpha}_2). \end{aligned} \tag{4.25}$$

The operators (4.24) also satisfy $a'_+a'_- + a_3'^2 = 0$ in analogy with the traceless condition (4.4). One may follow the analysis of § 2, and construct the highest-weight state $a_+'^l$, except that now equation (2.9) is allowed, since a legitimate square root is possible:

$$(a'_+)^{1/2} \to \alpha_1.$$

In this way the methods of § 2 are valid for both integral and half-integral l, and one has in fact reproduced the well known representation theory of SU(2). Such spinor representations are seen, therefore, to arise in a way that is completely different from the method involving traceless bosons. Such a construction has no relevance to orbital angular momentum, but rather to spin angular momentum, because the spinor $\alpha = (\alpha_1, \alpha_2)$ has no realizations in terms of space coordinates $\boldsymbol{x}$ and $\boldsymbol{\partial}$. One reason for this is that one cannot take the square root $(a_+)^{1/2}$, as is possible for the Cartan spinors.

Let us explain this important point in more detail. If one formulates the methods of § 2 using traceless bosons, as we have done, and tries to include the spinor representations by taking the square root, as shown in equation (2.9), then one must first explain the symbol $(a_+)^{1/2}$. We assert that, in contrast to the expression $(x_+)^{1/2}$, the operator $(a_+)^{1/2}$ is not defined. More generally, the operator

$$S = a_+^\lambda \tag{4.26}$$

is defined only for integral $\lambda \geqslant 0$. This can be seen in the following way. In any definition of $(a_+)^{1/2}$ it would be required that $(a_+)^{1/2} \cdot 1 = (x_+)^{1/2}$ and $(a_+)^{-1/2} \cdot 1 = (x_+)^{-1/2}$, this being merely the extension to half integers n of the equation

$$a_+^n \cdot 1 = x_+^n,$$

which is true for integers $n \geqslant 0$, as is proved by induction. Now,

$$\begin{aligned} (a_+)^{1/2} \cdot 1 &= a_+(a_+)^{-1/2} \cdot 1 \\ &= [x_+ - r^2\partial_+(2N+1)^{-1}](x_+)^{-1/2} \\ &= (x_+)^{1/2} - r^2(0/0), \end{aligned} \tag{4.27}$$

that is, the second term is not zero as consistency requires, but is indeterminate. Although indeterminate quantities will arise when physical operators such as momentum are introduced, in the present group theoretical context the indeterminate quantity in (4.27) is inconsistent and not legitimate. Hence $(a_+)^{1/2}$ is not defined and the indecomposable representations obtained in § 2 are eliminated right from the start.

The computation of the matrix elements (4.23) for $j=\frac{1}{2}$ is readily generalized to any representation label j. If the matrix elements of T_g are defined by

$$T_g\phi_{jm} = \sum_{m'=-j}^{j} [T_g]^{(j)}_{m'm}\phi_{jm'}$$

then these matrix elements, which are calculated from (4.10) using the normalized basis (4.18), are the familiar Wigner functions:

$$[T_g]^{(j)}_{m'm} = D^j_{m'm}(\alpha, \beta, \gamma). \tag{4.28}$$

In the original parametrization of g they appear as Jacobi polynomials. This derivation applies to both half-integral and integral values of j, whereas previously a proof was possible for integral j only. It was noticed by Schwinger (1965), and earlier by Bopp and Haag (1950) that the matrix elements (4.28) could indeed be consistently extended to the case of half-integral j, m and m', and in a similar context also by Goldberg *et al* (1967) for spin s spherical harmonics, and we have now proved this.

Let us remark on some eigenvalue properties. The half-integral eigenvalues arise because λ may take half-integral values, and this may be understood (although not proved) in a simple way. The commutation relations (4.13) are satisfied for any value of λ in (4.11), but λ is quantized because in the space of harmonic polynomials (from equation (4.11)),

$$J^2 = (N-\lambda)(N-\lambda+1).$$

N has only integral eigenvalues, so that λ takes half-integral or integral values. The eigenvalue of N, which is l, can take only integral values because its domain comprises only polynomials, and so from (4.14) we have that possible values of j, which must be non-negative, are either

$$j = 0, 1, 2, \ldots \qquad (\lambda \text{ integral}),$$

or

$$j = \tfrac{1}{2}, \tfrac{3}{2}, \tfrac{5}{2}, \ldots \qquad (\lambda \text{ half-integral}).$$

At this stage, however, there is nothing to connect the generators (4.11) with orbital angular momentum operators.

5. Equivalence mapping

We now point out an equivalence property which adds considerable insight as to the origin of the generators (4.11). For the moment let us restrict λ to integral values only, and observe that the operators J^2 and L^2 (as defined through equations (4.11) and (4.12)) have identical spectra; it might therefore be guessed that these operators are related by a similarity transformation S:

$$\boldsymbol{J} = S\boldsymbol{L}S^{-1}. \tag{5.1}$$

Under such a transformation the states ψ will be carried to $S\psi$. We therefore compare the basis states for $\lambda = 0$ given by (4.8) with those for general λ given by (4.16), and conjecture that

$$S = a_+^\lambda. \tag{5.2}$$

This operator does indeed accomplish the transformation (5.1), as is readily verified by transforming the representation mapping T_g. From (4.5)

$$T_g\psi(\boldsymbol{a}) = \psi(\boldsymbol{a}g),$$

so that under S,

$$T_g\psi(\boldsymbol{a}) \to a_+^\lambda T_g(a_+^{-\lambda}\psi'(\boldsymbol{a})) = (a_+/(\boldsymbol{a}g)_+)^\lambda \psi'(\boldsymbol{a}g),$$

in agreement with (4.10). Then (5.1) is verified, and shows in particular that the momentum and position operators are transformed to give

$$\boldsymbol{P} = -\mathrm{i}a_+^\lambda \boldsymbol{\partial} a_+^{-\lambda}, \tag{5.3}$$

$$\boldsymbol{Q} = a_+^\lambda \boldsymbol{x} a_+^{-\lambda}. \tag{5.4}$$

Although the presence of the equivalence mapping shows that the case of integral λ is of no interest, this situation changes completely for half-integral λ. In fact, for $\lambda = \frac{1}{2}$ the operator S does not exist, and the equivalence mapping (5.1) is no longer valid (as must be the case, considering that J^2 and L^2 have different spectra for $\lambda = \frac{1}{2}$). The fact that $(a_+)^{1/2}$ cannot be defined was shown in § 4, where S appeared (equation (4.26)) in a related context.

Using S, however, one can determine the structure of the generators $\boldsymbol{J}$ and define momentum and position operators, irrespective of the value of λ. This is because formal commutation relations are independent of the particular value of λ. The momentum $\boldsymbol{P}$ is evaluated as

$$\begin{aligned} P_j &= -\mathrm{i}\partial_j + \mathrm{i}[\partial_j, a_+^\lambda]a_+^{-\lambda} \\ &= -\mathrm{i}\partial_j + \mathrm{i}\frac{\lambda}{a_+}(\delta_{j1} + \mathrm{i}\delta_{j2}) - 2\mathrm{i}\lambda\frac{a_j}{a_+}\partial_+(2N - 2\lambda - 1)^{-1} \end{aligned} \tag{5.5}$$

(see appendix 2). We take this as the definition of $\boldsymbol{P}$ for $\lambda = \frac{1}{2}$, and similarly for $\boldsymbol{Q}$. These position and momentum operators will then automatically satisfy the canonical commutation relations. In addition, also because of the equivalence nature of the transformation S, the angular momentum $\boldsymbol{J}$ can be written $\boldsymbol{J} = \boldsymbol{Q} \times \boldsymbol{P}$, even for $\lambda = \frac{1}{2}$. This could, of course, be checked directly.

Accordingly, we can view $\boldsymbol{J}$ as the generators of orbital angular momentum, and the interpretation in terms of massless particles follows. We repeat, however, that the decomposition of $\boldsymbol{J}$ as $\boldsymbol{Q} \times \boldsymbol{P}$ is of a formal nature only and one observes, for example, that $\boldsymbol{P}$ has indeterminate matrix elements (appendix 2). Furthermore, although one can define an inner product using S, in order to obtain an Hermitian $\boldsymbol{P}$, the basis states would become non-normalizable.

For $\boldsymbol{J}$ to be a valid quantum mechanical observable it must be an Hermitian operator. This can be readily proved because in equation (4.10) we have the global form of these rotations, meaning that the integral forms of the generators are well defined. Therefore, let

$$U(\boldsymbol{t}) = \exp(\mathrm{i}\boldsymbol{t} \cdot \boldsymbol{J}), \tag{5.6}$$

and $U(\boldsymbol{t})$ is a family of unitary operators, because, with the inner product (4.17) we have defined unitary representations. Now, from (1.4) $U(\boldsymbol{t})$ satisfies

$$U(\boldsymbol{t}+\boldsymbol{s}) = \pm U(\boldsymbol{t})U(\boldsymbol{s}), \tag{5.7}$$

so that $U(\boldsymbol{t})$ does not form a family of unitary groups in the usual sense (Reed and Simon 1972, chap. 8). We have insisted here on strong continuity on the SO(3) group manifold, and so have identified a rotation by 2π with the identity transformation, thereby leading to the $\pm$ sign. To eliminate this, we let $\boldsymbol{t}$ and $\boldsymbol{s}$ be parameters on the SU(2) manifold, on which we again require continuity, and note that the representation must now be single-valued. The operator $U(\boldsymbol{t})$ then forms a family of unitary groups, and we can apply Stone's theorem (Reed and Simon 1972, chap. 8) to show that $\boldsymbol{J}$ are Hermitian operators.

6. Conclusion

We have described certain realizations of the rotation group in which half-integral angular momentum appears, despite the presence of scalar fields only. There is some formal similarity to the rotation group properties that appear in the Dirac monopole, but the interpretation is very different. We have interpreted the angular momentum here as purely orbital, and deduced that this can apply to massless particles only. We have examined in some detail the various properties of this realization, having analysed, for example, relevant spinors and their transformation properties. In particular, we have studied the double-valued nature of these half-integer representations.

In the full relativistic treatment it is expected that realizations of the Poincaré group will be found of a massless scalar particle taking all possible helicity values, including half integers. By application of the spin-statistics theorem, one would then have the new phenomenon of massless scalar fermions. Similar methods would be applicable, but without the problems originating from our non-relativistic treatment.

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Appendix 1

Let us indicate how to obtain the decomposition of $\boldsymbol{Q}$ into $\boldsymbol{Q}^{(\pm)}$ given in equations (3.6). According to Bracken and Green (1971) the projection operators $E^{(\pm)}$ with the property that $E^{(\pm)}\boldsymbol{Q} = \boldsymbol{Q}^{(\pm)}$ are given by

$$E^{(+)} = \frac{(\alpha+\Lambda)(\alpha-1)}{\Lambda(2\Lambda+1)}, \qquad E^{(-)} = \frac{(\alpha-\Lambda-1)(\alpha-1)}{(\Lambda+1)(2\Lambda+1)}.$$

Now, to evaluate $E^{(\pm)}\boldsymbol{Q}$ we use

$$\alpha\boldsymbol{Q} = -\mathrm{i}\boldsymbol{J}\times\boldsymbol{Q}, \qquad \alpha^2\boldsymbol{Q} = \Lambda(\Lambda+1)\boldsymbol{Q} - \mathrm{i}\boldsymbol{J}\times\boldsymbol{Q},$$

and equations (3.6) follow, except that we have discarded irrelevant normalizations.

Appendix 2

We calculate here the expression for $\boldsymbol{P}$ (equation (5.5)), and then verify that the conditions found in § 3, equation (3.7), are satisfied by both $\boldsymbol{Q}$ and $\boldsymbol{P}$.

Firstly we require

$$[\partial_j, a_+^k] = k a_+^{k-1}(\delta_{j1} + \mathrm{i}\delta_{j2}) - 2k a_j a_+^{k-1}\partial_+(2N + 2k - 1)^{-1},$$

for non-negative integers k. This equation follows by induction on k, using

$$[\partial_j, a_+] = \delta_{j1} + \mathrm{i}\delta_{j2} - 2a_j\partial_+(2N+1)^{-1}.$$

Then we have

$$-\mathrm{i}a_+^{\lambda}\partial_j a_+^{-\lambda} = -\mathrm{i}\partial_j + \frac{\mathrm{i}\lambda}{a_+}(\delta_{j1} + \mathrm{i}\delta_{j2}) - 2\mathrm{i}\lambda\frac{a_j}{a_+}\partial_+(2N - 2\lambda - 1)^{-1}$$

which equals P_j, as required. To find the matrix elements of $\boldsymbol{P}$, we obtain firstly the following, where ψ_{jm} is defined by equation (4.16):

$$a_+\psi_{jm} = \psi_{j+1,m+1},$$

$$a_-\psi_{jm} = -\psi_{j+1,m-1},$$

$$a_3\psi_{jm} = -\psi_{j+1,m},$$

and

$$\partial_+\psi_{jm} = -\frac{(j-m)(j-m-1)}{2j+2\lambda-1}\psi_{j-1,m+1},$$

$$\partial_-\psi_{jm} = \frac{(j+m+2\lambda)(j+m+2\lambda-1)}{2j+2\lambda-1}\psi_{j-1,m-1},$$

$$\partial_3\psi_{jm} = -\frac{(j-m)(j+m+2\lambda)}{2j+2\lambda-1}\psi_{j-1,m}.$$

The proof is by induction on j and m, and uses the commutation relations (4.3). The matrix elements of $\boldsymbol{P}$ are then calculated to be:

$$P_+\psi_{jm} = \frac{\mathrm{i}(j-m)(j-m-1)}{2j-1}\psi_{j-1,m+1},$$

$$P_-\psi_{jm} = -\frac{\mathrm{i}(j+m)(j+m-1)}{2j-1}\psi_{j-1,m-1},$$

$$P_3\psi_{jm} = \frac{\mathrm{i}(j-m)(j+m)}{2j-1}\psi_{j-1,m}.$$

We see that for $j = \frac{1}{2}$ these matrix elements are indeterminate, as is expected for a massless particle. We can now verify that $\boldsymbol{P}^{(-)}\psi_{1/2} = 0$. The matrix elements of $\boldsymbol{P}^{(-)}$ are given by

$$[(\Lambda+1)P_+ + \mathrm{i}(\boldsymbol{J}\times\boldsymbol{P})_+]\psi_{jm} = \mathrm{i}(j-m)(j-m-1)\psi_{j-1,m+1},$$

$$[(\Lambda+1)P_- + \mathrm{i}(\boldsymbol{J}\times\boldsymbol{P})_-]\psi_{jm} = -\mathrm{i}(j+m)(j+m-1)\psi_{j-1,m-1},$$

$$[(\Lambda+1)P_3 + \mathrm{i}(\boldsymbol{J}\times\boldsymbol{P})_3]\psi_{jm} = \mathrm{i}(j+m)(j-m)\psi_{j-1,m}.$$

Putting $j=\frac{1}{2}$, $m=\pm\frac{1}{2}$ we get $\boldsymbol{P}^{(-)}\psi_{1/2}=0$. A more complicated calculation is needed to show that $\boldsymbol{Q}^{(-)}\psi_{1/2}=0$, and uses the relation

$$\boldsymbol{a}=\boldsymbol{Q}-\mathrm{i}Q^2\boldsymbol{P}(2N-2\lambda+1)^{-1},$$

which follows from (4.1), (5.3), (5.4). Using now

$$[(\Lambda+1)\boldsymbol{a}+\mathrm{i}\boldsymbol{J}\times\boldsymbol{a}]\psi_{jm}=0,$$

it follows that $\boldsymbol{Q}^{(-)}\psi_{1/2}=0$.

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